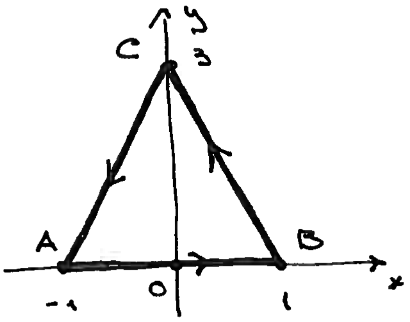
Correction de l'examen du 27/04/2012

Ex. 1



1). Calculons l'intégrale

$$\int_\gamma y^2\,dx + x\,dy$$

Paramétrisation de AB :

$$\begin{cases} x(t) = t & \Rightarrow dx = dt \\ y(t) = 0 & \Rightarrow dy = 0 \end{cases} \qquad t \in [-1,1]$$

$$\int_{AB} y^2\,dx + x\,dy = \int_{-1}^{1} 0\cdot dt + t\cdot 0 = 0$$

Paramétrisation de BC :

$$\begin{cases} x(t) = 1\,(1-t) + 0\cdot t = 1-t & \Rightarrow dx = -dt \\ y(t) = 0\cdot(1-t) + 3\cdot t = 3t & \Rightarrow dy = 3\,dt \\ t \in [0,1] \end{cases}$$

$$\int_{BC} y^2dx + x\,dy = \int_0^1 (3t)^2(-1)\,dt + (1-t)\cdot 3\,dt =$$

$$= 3\int_0^1 (1 - t - 3t^2)\,dt = 3\left[t - \frac{t^2}{2} - t^3\right]_0^1 = -\frac{3}{2}$$

Paramétrisation de CA :

$$\begin{cases} x(t) = 0(1-t) + (-1)\,t = -t & \Rightarrow dx = -dt \\ y(t) = 3(1-t) + 0\cdot t = 3(1-t) & \Rightarrow dy = -3\,dt \\ t \in [0,1] \end{cases}$$

$$\int_{CA} y^2 dx + x\,dy = \int_0^1 9(1-t)^2\cdot(-)\,dt + (-t)(-3)\,dt =$$

$$= 3\int_0^1 (-3 + 7t - 3t^2)\,dt = 3\left[-3t + \frac{7t^2}{2} - t^3\right]_0^1 = -\frac{3}{2}$$

Donc au final $\int_\gamma y^2dx + x\,dy = -\frac{3}{2} - \frac{3}{2} = -3.$

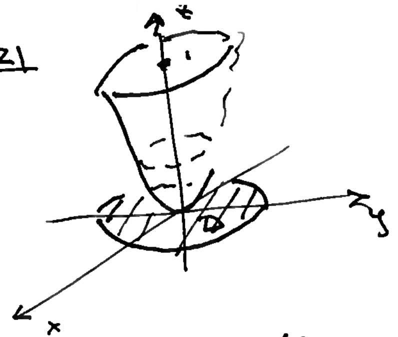
2). D'après le théorème de Green

$$\int_\gamma y^2 dx + x\,dy = \iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial y^2}{\partial y}\right) dx\,dy = \iint_D (1-2y)\,dx\,dy =$$

$$= \text{aire}(D) - 2\iint_D y\,dy = 3 - 2\iint_D y\,dy = 3 - 4\int_0^1 dx \int_0^{3-3x} y\,dy =$$

$$= 3 - 4\int_0^1 dx\; \frac{9(1-x)^2}{2} = 3 - 18\left[\frac{(x-1)^3}{3}\right]_0^1 = 3 - 6 = -3$$

Ex. 2)



Paramétrisation de S :

$$\begin{cases} x = u \\ y = v \\ z = u^2 + v^2 \\ (u,v) \in D \end{cases}$$

Alors $\vec{r}\,'_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}$, $\vec{r}\,'_v = \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix}$, $\vec{r}\,'_u \wedge \vec{r}\,'_v = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}$,

$$|\vec{r}\,'_u \wedge \vec{r}\,'_v| = \sqrt{1+4(u^2+v^2)}$$

L'aire de S s'écrit donc comme

$$\text{aire}(S) = \iint_D \sqrt{1+4(u^2+v^2)}\, du\,dv = \left| \begin{array}{l} u = r\cos\varphi \\ v = r\sin\varphi \\ du\,dv \to r\,dr\,d\varphi \\ \varphi \in [0,2\pi],\ r \in [0,1] \end{array} \right| =$$

$$= \int_0^1 \sqrt{1+4r^2}\, r\,dr \int_0^{2\pi} d\varphi = 2\pi \int_0^1 r\sqrt{1+4r^2}\,dr = \left| \begin{array}{l} 1+4r^2 = t \\ dt = 8r\,dr \end{array} \right| =$$

$$= \frac{1}{8} \cdot 2\pi \cdot \int \sqrt{t}\,dt = \frac{\pi}{4} \frac{t^{3/2}}{3/2} = \frac{\pi}{6} \left\{ [1+4r^2]^{3/2} \right\}_{r=0}^{1} = \frac{\pi}{6}(5\sqrt{5}-1).$$

Ex. 3) $f(z) = \dfrac{1-\cos z}{z \sin z}$

Points candidats : solutions de $z\sin z = 0 \Rightarrow z = \pi k,\ k \in \mathbb{Z}$

$$z = 0 \Rightarrow f(z) \sim \frac{1 - \left(1 - \frac{z^2}{2} + \dots\right)}{z \cdot (z + \dots)} \xrightarrow{z \to 0} \frac{1}{2} \Rightarrow \text{il n'y a pas de pôle}$$

$$z = \pi k,\ k \neq 0 \Rightarrow \sin z = [\sin z]_{z=\pi k} + [\cos z]_{z=\pi k}(z-\pi k) + \dots =$$

$$= (-1)^k (z - \pi k) + \dots$$

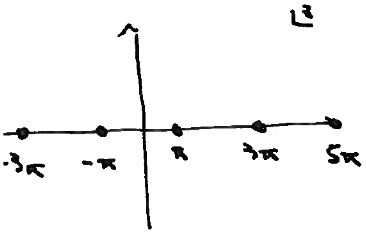


Fig. 1. Pôles de f

$$1 - \cos z = [1-\cos z]_{z=\pi k} + [\sin z]_{z=\pi k}(z-\pi k) + \dots$$

$$= [1 - (-1)^k] + \underset{\text{inconnu}}{\text{coeg.}}\,(z-\pi k)^2 + \dots$$

Donc pour k paires $\Rightarrow$ pas de pôle

k impaires $\Rightarrow$ pôle simple

Ex. 4) 1. $$\int_{-\infty}^{\infty} \frac{x^2+x+1}{(x^2+4)(x^2+9)}\,dx = \int_{\underset{R\to\infty}{\curvearrowright}} \frac{z^2+z+1}{(z^2+4)(z^2+9)}\,dz = \left| \begin{array}{l} \text{d'après le thm} \\ \text{des résidus} \end{array} \right| =$$

$$= 2\pi i \sum_{\substack{\text{pts sing.} \\ \text{avec } \operatorname{Im} z > 0}} \operatorname{res} \frac{z^2+z+1}{(z^2+4)(z^2+9)} = 2\pi i \left\{ \operatorname{res}_{z=2i} \frac{z^2+z+1}{(z-2i)(z+2i)(z^2+9)} + \operatorname{res}_{z=3i} \frac{z^2+z+1}{(z^2+4)(z-3i)(z+3i)} \right\}$$

$$= 2\pi i \left\{ \frac{(2i)^2+2i+1}{4i\,(-4+9)} + \frac{(3i)^2+3i+1}{(-9+4)\cdot 6i} \right\} = 2\pi i \left\{ \frac{-3+2i}{20i} - \frac{-8+3i}{30i} \right\} =$$

$$= \frac{2\pi i}{i} \left(\frac{8}{30} - \frac{3}{20} \right) = 2\pi \cdot \frac{7}{60} = \frac{7\pi}{30}.$$

21. $$\int_0^{2\pi} \frac{d\theta}{(3-\cos\theta)^2} = \left| \begin{array}{l} z = e^{i\theta} \\ \cos\theta = \frac{z+z^{-1}}{2} \\ d\theta = \frac{dz}{iz} \end{array} \right| = \oint_{|z|=1} \frac{dz}{iz\left(3 - \frac{z+z^{-1}}{2}\right)^2} =$$

$$= \frac{4}{i} \oint_{|z|=1} \frac{z\,dz}{(6z-z^2-1)^2} = \frac{4}{i} \oint_{|z|=1} \frac{z\,dz}{(z^2-6z+1)^2}$$

Solutions de $z^2-6z+1=0$ $\Rightarrow$ pôles de 2nd ordre

$$\Delta = 36-4 = 32 = (4\sqrt{2})^2$$

$$z = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

Seul pôle à l'intérieur de $|z|=1$ est $z = 3-2\sqrt{2}$, donc

$$\Leftrightarrow 2\pi i \cdot \frac{4}{i} \operatorname{res} \frac{z}{(z-[3+2\sqrt{2}])^2 (z-[3-2\sqrt{2}])^2} =$$

$$= 8\pi \left[\frac{z}{(z-[3+2\sqrt{2}])^2} \right]'_{z=3-2\sqrt{2}} = 8\pi \left\{ \frac{1}{(z-[3+2\sqrt{2}])^2} - \frac{2z}{(z-[3+2\sqrt{2}])^3} \right\}_{z=3-2\sqrt{2}}$$

$$= 8\pi \left\{ \frac{1}{(-4\sqrt{2})^2} - \frac{2\cdot(3-2\sqrt{2})}{(-4\sqrt{2})^3} \right\} = 8\pi \cdot \frac{6}{(4\sqrt{2})^3} = 16 \cdot \frac{3\pi}{128\sqrt{2}} =$$

$$= \frac{3\pi}{8\sqrt{2}}.$$